Assignment 10

Coverage: 16.4 in Text.

Optional reading: Wiki "Green's theorem", "cross product".

Exercises: 16.4 no 7, 11, 14, 23, 26, 27, 30, 37, 39.

Hand in 16.4 no 14, 27, 39 by April 13.

Supplementary Problems

1. A vector field $\mathbf{F}$ is called radial if $\mathbf{F}(x, y, z) = f(r)(x, y, z),\ r = \sqrt{x^2 + y^2 + z^2}$, for some function f. Show that every radial vector field is conservative. You may assume it is C^1 in $\mathbb{R}^3$.

2. Let $F = (P, Q)$ be a C^1-vector field in $\mathbb{R}^2$ away from the origin. Suppose that $P_y = Q_x$. Show that for any simple closed curve C enclosing the origin and oriented in positive direction, one has
$$\oint_C Pdx + Qdy = \lim_{\varepsilon \to 0} \varepsilon \int_0^{2\pi} [-P(\varepsilon \cos\theta, \varepsilon \sin\theta) \sin\theta + Q(\varepsilon \cos\theta, \varepsilon \sin\theta) \cos\theta]\, d\theta \ .$$
What happens when C does not enclose the origin?

3. (Optional) We identity the complex plane with $\mathbb{R}^2$ by $x + iy \mapsto (x, y)$. A complex-valued function f has its real and imaginary parts respectively given by $u(x, y) = Ref(z)$ and $v(x, y) = Imf(z)$. Note that u and v are real-valued functions. The function f is called differentiable at z if
$$\frac{df}{dz}(z) = \lim_{w \to 0} \frac{f(z + w) - f(z)}{w} \ ,$$
exists.

   (a) Show that f is differentiable at z implies that the partial derivatives of u and v exist and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, hold. Hint: Take $w = h, ih$, where $h \in \mathbb{R}$ and then let $h \to 0$.

   (b) Propose a definition of $\int_C f\, dz$, where C is an oriented curve in the plane, in terms of the line integrals involving u and v.

   (c) Suppose that f is differentiable everywhere in $\mathbb{C}$. Show that for every simple closed curve C,
$$\oint_C f\, dz = 0 \ .$$